

Global well-posedness for the two-dimensional equations of nonhomogeneous incompressible liquid crystal flows with nonnegative density^{*}

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Abstract. In this paper, the authors first establish the global well-posedness of strong solutions of the simplified Ericksen-Leslie model for nonhomogeneous incompressible nematic liquid crystal flows in two dimensions if the initial data satisfies some smallness condition. It is worth pointing out that the initial density is allowed to contain vacuum states and the initial velocity can be arbitrarily large. We also present a Serrin's type criterion, depending only on ∇d , for the breakdown of local strong solutions. As a byproduct, the global strong solutions with large initial data are obtained, provided the macroscopic molecular orientation of the liquid crystal materials satisfies a natural geometric angle condition (cf. [19]).

Keywords. Liquid crystals; Nonhomogeneous incompressible flows; Global strong solutions; Vacuum; Blowup criterion

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1 Introduction

Liquid crystals are substances that exhibit a phase of matter that has properties between those of a conventional liquid and those of a solid crystal (cf. [12]). The hydrodynamic theory of liquid crystals was first developed by Ericksen and Leslie during the period of 1958 through 1968 (see [9, 10, 20, 21]). Since then, many remarkable developments have been made from both theoretical and applied aspects, however, many physically important and mathematically fundamental problems still remain open. In this paper, we consider a simplified Ericksen-Leslie model for the nonhomogeneous incompressible nematic liquid crystals in two dimensions:

$$\rho_t + \operatorname{div}(\rho u) = 0, \quad (1.1)$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = \Delta u - \nabla d \cdot \Delta d, \quad (1.2)$$

$$\operatorname{div} u = 0, \quad (1.3)$$

$$d_t + u \cdot \nabla d = \Delta d + |\nabla d|^2 d, \quad (1.4)$$

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where $\rho : \mathbb{R}^2 \times [0, \infty) \rightarrow \mathbb{R}^+$ is the density of the fluid, $u : \mathbb{R}^2 \times [0, \infty) \rightarrow \mathbb{R}^2$ is the velocity field of the fluid, $P : \mathbb{R}^2 \times [0, \infty) \rightarrow \mathbb{R}$ is the pressure of the fluid, and $d : \mathbb{R}^2 \times [0, \infty) \rightarrow \mathbb{S}^2$ (the unit sphere in \mathbb{R}^3 , i.e. $|d| = 1$) represents the averaged macroscopic/continuum molecular orientations.

Though system (1.1)–(1.4) is a simplified version of the Ericksen-Leslie model, but it still retains the most interesting mathematical properties without losing the basic nonlinear structure of the original Ericksen-Leslie model [9, 10, 20, 21]. Roughly speaking, the system (1.1)–(1.4) is a system of the nonhomogeneous Navier-Stokes equations for incompressible flows coupled with the equation for heat flow of harmonic maps, and thus, its mathematical analysis is full of challenges. In particular, if $\rho = \text{Const.}$, then it turns into the following homogeneous system which models the incompressible flows of nematic liquid crystal

$$u_t + u \cdot \nabla u + \nabla P = \mu \Delta u - \nabla d \cdot \Delta d, \quad (1.5)$$

$$\text{div} u = 0, \quad (1.6)$$

$$d_t + u \cdot \nabla d = \Delta d + |\nabla d|^2 d \quad (1.7)$$

with $|d| = 1$. Moreover, if $u = 0$ in (1.5)–(1.7), then it reduces to the following equation for heat flow of harmonic maps:

$$d_t = \Delta d + |\nabla d|^2 d, \quad |d| = 1. \quad (1.8)$$

There has been a lot of literature on the mathematical studies of (1.5)–(1.7) and (1.8), see, for example, [14, 13, 23, 24, 25, 26, 33, 35] and [3, 5, 4, 6, 32], respectively. In the following, we briefly recall some related mathematical results of the liquid crystal flows. In a series of papers, Lin [23] and Lin-Liu [24, 25] initiated the mathematical analysis of (1.5)–(1.7) in 1990s. More precisely, to relax the nonlinear constraint $|d| = 1$, they proposed an approximate model of Ericksen-Leslie system with variable length by Ginzburg-Landau functionals, that is, the equation (1.7) with $|d| = 1$ is replaced by

$$d_t + u \cdot \nabla d = \Delta d + \frac{1}{\varepsilon^2} (1 - |d|^2) d. \quad (1.9)$$

In [23, 24], the authors proved the global existence of classical and weak solutions of (1.5), (1.6), (1.9) in dimensions two and three, respectively. The partial regularity of suitable weak solutions was also studied in [25]. However, as pointed out in [24], the vanishing limit of $\varepsilon \rightarrow 0$ is an open and challenging problem. Indeed, in contrast with (1.9), it is much more difficult to deal with the nonlinear term $|\nabla d|^2 d$ with $|d| = 1$ appearing on the right-hand side of (1.4) or (1.7) from the mathematical point of view. In two independent papers [13] and [26], Hong and Lin-Lin-Wang showed the global existence of weak solutions of (1.5)–(1.7) in dimensions two, and proved that the solutions are smooth away from at most finitely many singular times which is analogous to that for the heat flows of harmonic maps (see [3, 32]). The global existence of smooth solution with small initial data of (1.5)–(1.7) was also proved [26, 33] and [35, 22] in dimensions two and three, respectively.

For the approximate nonhomogeneous equations (1.1)–(1.3) and (1.9), the global existence of weak solutions with generally large initial data was proved in [28, 16], and the global regularity of the solution with strictly positive density was studied in [8]. As aforementioned, the nonlinear term $|\nabla d|^2 d$ with $|d| = 1$ will cause serious difficulty in the mathematical analysis of liquid crystal flows. Recently, Wen and Ding [34] established the local existence and uniqueness of strong solutions of (1.1)–(1.4) in the case that the initial density may contain vacuum states (i.e. $\rho_0 \geq 0$). Moreover, if the initial density has a positive lower bound (i.e. $\rho_0 \geq \underline{\rho} > 0$) which

indicates that there is absent of vacuum initially, the global strong solutions with small initial data was also obtained in [34].

As that for the density-dependent Navier-Stokes equations (see [7, 27]), the possible presence of vacuum is one of the major difficulties when the problems of global existence, uniqueness and regularity of solutions are involved. Therefore, in the present paper we aim to investigate the global regularity of (1.1)–(1.4) when the initial density may contain vacuum.

We consider the Cauchy problem of (1.1)–(1.4) with the following initial data:

$$(\rho, u, d)(x, 0) = (\rho_0, u_0, d_0)(x) \quad \text{for } x \in \mathbb{R}^2, \quad (1.10)$$

and the far-field behavior at infinity:

$$(\rho, u, d)(x, t) \rightarrow (\tilde{\rho}, 0, e) \quad \text{as } |x| \rightarrow \infty, \quad t > 0, \quad (1.11)$$

where $\tilde{\rho} > 0$ is a given positive constant and $e \in \mathbb{S}^2$ is a given unit vector (i.e. $|e| = 1$).

To state our main results, we first introduce the definition of strong solutions of (1.1)–(1.4), (1.10) and (1.11).

Definition 1.1 A pair of functions (ρ, u, P, d) is called a strong solution of (1.1)–(1.4), (1.10) and (1.11) on $\mathbb{R}^2 \times [0, T]$, if $\rho(x, t) \geq 0$ for all $(x, t) \in \mathbb{R}^2 \times [0, T]$,

$$\left\{ \begin{array}{l} \rho - \tilde{\rho} \in C([0, T]; H^2(\mathbb{R}^2)), \quad \rho_t \in L^\infty(0, T; H^1(\mathbb{R}^2)) \\ u \in C([0, T]; H^2(\mathbb{R}^2)) \cap L^2(0, T; H^3(\mathbb{R}^2)), \\ \sqrt{\rho} u_t \in L^\infty(0, T; L^2(\mathbb{R}^2)), \quad u_t \in L^2(0, T; H^1(\mathbb{R}^2)), \\ \nabla P \in C([0, T]; L^2(\mathbb{R}^2)) \cap L^2(0, T; H^1(\mathbb{R}^2)), \\ \nabla d \in C([0, T]; H^2(\mathbb{R}^2)), \quad d_t \in L^\infty(0, T; H^1(\mathbb{R}^2)) \cap L^2(0, T; H^2(\mathbb{R}^2)), \end{array} \right. \quad (1.12)$$

and (ρ, u, P, d) satisfies (1.1)–(1.4) a.e. on $\mathbb{R}^2 \times (0, T]$.

Then, our first result concerning the global strong solutions with small data can be stated in the following theorem.

Theorem 1.1 Assume that the initial data (ρ_0, u_0, d_0) satisfies

$$\left\{ \begin{array}{l} \rho_0 \geq 0, \quad (\rho_0 - \tilde{\rho}, u_0, \nabla d) \in H^2(\mathbb{R}^2), \quad \operatorname{div} u_0 = 0, \quad |d_0| = 1, \\ \Delta u_0 - \nabla P_0 - \nabla d_0 \cdot \Delta d_0 = \rho_0^{1/2} g \quad \text{for some } (\nabla P_0, g) \in L^2(\mathbb{R}^2). \end{array} \right. \quad (1.13)$$

Then for any given $0 < T < \infty$, there exists a unique global strong solution (ρ, u, P, d) of (1.1)–(1.4), (1.10) and (1.11) on $\mathbb{R}^2 \times [0, T]$, provided

$$\exp \left(2 \left(\|\rho_0^{1/2} u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2 \right) \right) \|\nabla d_0\|_{L^2}^2 \leq \frac{1}{16}. \quad (1.14)$$

It is worth mentioning that the smallness condition (1.14) stated in Theorem 1.1 implies that (ρ_0, u_0) can be arbitrarily large if $\|\nabla d_0\|_{L^2}$ is chosen to be suitably small. This is analogous to the one in [33]. Moreover, as a result, we see that the strong solution to the Cauchy problem of nonhomogeneous Navier-Stokes equations (i.e. $d = \text{Const.}$) with large initial data, which may contain vacuum, exists globally on $\mathbb{R}^2 \times [0, T]$ for all $0 < T < \infty$. Thus, Theorem 1.1 also generalizes the result due to Huang-Wang [14].

The proof of Theorem 1.1 is mainly based on a critical Sobolev inequality of logarithmic type which was recently proved by Huang-Wang (cf. [15]) and is originally due to Brezis-Wainger [2] (see also [29, 31]). However, it is remarkable that the arguments in [15] actually depend on the size of the domain considered and cannot be applied directly to the case of the whole space. Thus, some new ideas have to be developed. The main difference lies in the proof of Lemma 3.3, where, instead of $\|\rho^{1/2}u_t\|_{L^2}$ and $\|\rho^{1/2}u \cdot \nabla u\|_{L^2}$, we use the material derivative $\|\rho^{1/2}\dot{u}\|_{L^2}$ for some technical reasons. We also note here that the strictly positive far-field condition $\bar{\rho} > 0$ plays an important role in our analysis. The strongly nonlinear terms $|\nabla d|^2 d$ and $\nabla d \cdot \Delta d$ in (1.2) and (1.4) will also cause some additional difficulties.

For the generally large initial data, it is still an interesting and open problem whether the strong solution blows up or not in finite time. In [26] and [14], the authors proved respectively that the following blowup criteria for the two-dimensional equations of (1.5)–(1.7):

$$\lim_{T \rightarrow T^*} \int_0^T (\|u\|_{L^4}^4 + \|\nabla d\|_{L^4}^4) dt = \infty \quad \text{and} \quad \lim_{T \rightarrow T^*} \int_0^T \|\nabla d\|_{L^\infty} dt = \infty, \quad (1.15)$$

where $0 < T^* < \infty$ is the maximal time of the existence of a strong solution to (1.5)–(1.7). Motivated by the proofs of Theorem 1.1, we can prove the following mechanism for possible breakdown of strong solutions, which is a natural extension of the ones in [26, 14].

Theorem 1.2 *Assume that $0 < T^* < \infty$ is the maximal time of the existence of a strong solution to (1.1)–(1.4), (1.10) and (1.11) with generally large initial data (ρ_0, u_0, d_0) satisfying (1.13). Then,*

$$\lim_{T \rightarrow T^*} \int_0^T \|\nabla d\|_{L^r}^s dt = \infty \quad (1.16)$$

for any (r, s) satisfying

$$\frac{1}{r} + \frac{1}{s} \leq \frac{1}{2}, \quad 2 < r \leq \infty. \quad (1.17)$$

Theorem 1.2 implies that for any $0 < T < \infty$ if the left-hand side of (1.16) is finite, then the strong solution of (1.1)–(1.4), (1.10) and (1.11) will exist globally on $\mathbb{R}^2 \times (0, T)$.

Based on a frequency localization argument combined with the concentration-compactness approach, Lei-Li-Zhang [19] recently proved the following interesting rigidity theorem for the approximate harmonic maps.

Proposition 1.1 *([19, Theroem 1.5]) For given positive constants $0 < C_0 < \infty$ and $0 < \varepsilon \leq 1$, assume that $d : \mathbb{R}^2 \rightarrow \mathbb{S}^2$ satisfying $\nabla d \in H^1(\mathbb{R}^2)$ with $\|\nabla d\|_{L^2} \leq C_0$ and $d_3 \geq \varepsilon$. Then there exists a positive constant $\delta_0 \in (0, 1)$, which depends only on C_0 and ε , such that*

$$\|\nabla d\|_{L^4}^4 \leq (1 - \delta_0) \|\nabla^2 d\|_{L^2}^2, \quad (1.18)$$

which particularly implies

$$\|\Delta d + |\nabla d|^2 d\|_{L^2}^2 \geq \frac{\delta_0}{2} (\|\Delta d\|_{L^2}^2 + \|\nabla d\|_{L^4}^4). \quad (1.19)$$

As an immediate consequence of Theorem 1.2 and Proposition 1.1, we can remove the smallness restriction (1.14) on the initial data and prove the following existence theorem of global strong solutions with large initial data, provided the macroscopic molecular orientation of the liquid crystal materials satisfies a natural geometric angle condition. This extends the Lei-Li-Zhang's result (cf. [19]) to the case of nonhomogeneous incompressible liquid crystal flows with initial vacuum.

Theorem 1.3 *Let $e_3 = (0, 0, 1) \in \mathbb{S}^2$ and let d_{03} be the third component of d_0 . Besides the condition (1.13) in Theorem 1.1, assume further that*

$$d_{03} \geq \varepsilon \quad \text{and} \quad d_0 - e_3 \in L^2(\mathbb{R}^2) \quad (1.20)$$

holds for some uniform positive constant $\varepsilon > 0$. Then for any $0 < T < \infty$, there exists a unique global strong solution (ρ, u, P, d) of (1.1)–(1.4), (1.10) and (1.11) on $\mathbb{R}^2 \times [0, T]$.

The rest of the paper is organized as follows. In Sect. 2, we state some known inequalities and facts which will be used later. The proof of Theorem 1.1 will be done in Sect. 3, based on the local existence theorem and the global a priori estimates. In Sect. 4, we outline the proof of Theorems 1.2 and 1.3.

2 Preliminaries

In this section, we list some useful lemmas which will be frequently used in the next sections. We first recall the well-known Ladyzhenskaya and Sobolev inequalities (see, for example, [17, 1]).

Lemma 2.1 *For $f \in H^1(\mathbb{R}^2)$, it holds for any $2 \leq p < \infty$ that*

$$\|f\|_{L^4}^2 \leq \sqrt{2} \|f\|_{L^2} \|\nabla f\|_{L^2}, \quad (2.1)$$

$$\|f\|_{L^p} \leq C(p) \|f\|_{L^2}^{2/p} \|\nabla f\|_{L^2}^{1-2/p}, \quad (2.2)$$

where $C(p)$ is a positive constant depending on p . In addition, if $f \in W^{1,p}(\mathbb{R}^2) \cap H^2(\mathbb{R}^2)$ with $p > 2$, then there exists a universal positive constant C such that

$$\|f\|_{L^\infty} \leq C \|f\|_{W^{1,p}} \leq C \|f\|_{H^2}. \quad (2.3)$$

We will also use the following Poincaré type inequality, which shows that the velocity u actually belongs to L^2 -space even that the vacuum states may appear.

Lemma 2.2 *Let $\tilde{\rho} > 0$ be a given positive constants. Assume that $\varrho - \tilde{\rho} \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with $\varrho(x) \geq 0$, $\nabla v \in L^2(\mathbb{R}^2)$ and $\sqrt{\varrho}v \in L^2(\mathbb{R}^2)$. Then,*

$$\|v\|_{L^2} \leq C(\tilde{\rho}, \|\varrho - \tilde{\rho}\|_{L^2 \cap L^\infty}) \left(\|\varrho^{1/2}v\|_{L^2} + \|\nabla v\|_{L^2} \right), \quad (2.4)$$

where $C(\tilde{\rho}, \|\varrho - \tilde{\rho}\|_{L^2 \cap L^\infty})$ is a positive constant depending only on $\tilde{\rho}$, $\|\varrho - \tilde{\rho}\|_{L^2}$ and $\|\varrho - \tilde{\rho}\|_{L^\infty}$.

Proof. Indeed, by virtue of Hölder and (2.2), we have for any $q \geq 2$ that

$$\begin{aligned} \tilde{\rho} \int |v|^2 dx &= \int \varrho |v|^2 dx - \int (\varrho - \tilde{\rho}) |v|^2 dx \\ &\leq C \|\varrho^{1/2}v\|_{L^2}^2 + C \left(\int |\varrho - \tilde{\rho}|^q dx \right)^{1/q} \left(\int |v|^{2q/(q-1)} dx \right)^{(q-1)/q} \\ &\leq C(\tilde{\rho}, \|\varrho - \tilde{\rho}\|_{L^2 \cap L^\infty}) \left(\|\varrho^{1/2}v\|_{L^2}^2 + \|v\|_{L^2}^{2(q-1)/q} \|\nabla v\|_{L^2}^{2/q} \right) \\ &\leq C(\tilde{\rho}, \|\varrho - \tilde{\rho}\|_{L^2 \cap L^\infty}) \left(\|\varrho^{1/2}v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \right) + \frac{\tilde{\rho}}{2} \|v\|_{L^2}^2, \end{aligned}$$

which proves (2.4) immediately. \square

Next, to improve the regularity of the velocity, we need to use the following estimates of the Stokes equations (see, for example, [11, 18]).

Lemma 2.3 *Consider the following stationary Stokes equations:*

$$-\Delta U + \nabla P = f, \quad \operatorname{div} U = 0 \quad \text{in } \mathbb{R}^2.$$

Then for any $f \in W^{m,p}(\mathbb{R}^2)$ with $m \in \mathbb{Z}^+$ and $p > 1$, there exists a positive constant C , depending only on m and p , such that

$$\|\nabla^2 U\|_{W^{m,p}} + \|\nabla P\|_{W^{m,p}} \leq C\|f\|_{W^{m,p}}. \quad (2.5)$$

To estimate the L^2 -norm of the gradient of the velocity, we shall apply a critical Sobolev inequality of logarithmic type which was proved by Huang-Wang (cf. [15]) and is originally due to Brezis-Wainger [2] (see also [29, 31]). This is the key tool for the proofs of Theorems 1.1–1.3.

Lemma 2.4 *For $q > 2$ and $0 \leq s < t < \infty$, assume that $f \in L^2(s, t; H^1(\mathbb{R}^2)) \cap L^2(s, t; W^{1,q}(\mathbb{R}^2))$. Then there exists a positive constant $C(q)$, independent of s, t , such that*

$$\|f\|_{L^2(s, t; L^\infty(\mathbb{R}^2))} \leq C \left(1 + \|f\|_{L^2(s, t; H^1(\mathbb{R}^2))} (\ln^+ \|f\|_{L^2(s, t; W^{1,q}(\mathbb{R}^2))})^{1/2} \right). \quad (2.6)$$

In the case that the lower bound of the density is nonnegative, the local existence of strong solutions to (1.1)–(1.4), (1.10) and (1.11) was proved in [34]. Indeed, in [34] the authors only considered the case of smooth bounded domains, however, as pointed out in [7], the similar procedure also works for the whole space by means of the standard domain expansion technique. For simplicity, we quote the following local existence theorem of strong solutions without proofs.

Lemma 2.5 *Assume that the conditions of Theorem 1.1 hold. Then there exists a positive time $0 < T_0 < \infty$ such that the Cauchy problem (1.1)–(1.4), (1.10) and (1.11) admits a unique strong solution on $\mathbb{R}^2 \times (0, T_0)$.*

3 Proof of Theorem 1.1

Assume that the conditions of Theorem 1.1 hold. Let $0 < T^* < \infty$ be the first blowup time of a strong solution (ρ, u, P, d) to the Cauchy problem (1.1)–(1.4), (1.10) and (1.11). In order to prove Theorem 1.1, it suffices to prove there actually exists a generic positive constant $0 < M < \infty$, depending only on the initial data (ρ_0, u_0, d_0) and T^* , such that

$$\begin{aligned} \mathcal{E}(T) \triangleq & \sup_{0 \leq t \leq T} \left(\|\rho - \tilde{\rho}\|_{H^2} + \|u\|_{H^2} + \|\nabla d\|_{H^2} + \|\rho^{1/2} u_t\|_{L^2}^2 + \|d_t\|_{H^1}^2 \right) \\ & + \int_0^T (\|u\|_{H^3}^2 + \|\nabla d\|_{H^3}^2 + \|u_t\|_{H^1}^2 + \|d_t\|_{H^2}^2) dt \leq M \end{aligned} \quad (3.1)$$

holds for any $0 < T < T^*$. So, by the local existence theorem (see Lemma 2.5) it can be easily shown that the strong solution can be extended beyond T^* , which gives a contradiction of T^* . Hence, the strong solution exists globally on $\mathbb{R}^2 \times [0, T]$ for any $0 < T < \infty$. The proof of Theorem 1.1 is therefore complete.

The proof of (3.1) is based on a series of lemmas. Throughout the remainder of the paper, for simplicity we denote by C a generic constant which depends only on the initial data and T^* , and may change from line to line.

First, it is easy to see from the method of characteristics and (1.1) that for every $0 < T < T^*$,

$$0 \leq \rho(x, t) \leq \|\rho_0\|_{L^\infty} \quad \text{for all } (x, t) \in \mathbb{R}^2 \times [0, T]. \quad (3.2)$$

Moreover, multiplying (1.1) by $q|\rho - \tilde{\rho}|^{q-2}(\rho - \tilde{\rho})$ with $q \geq 2$, integrating it by parts over $(0, t)$, and using the divergence-free condition (1.3), we find that

$$\|(\rho - \tilde{\rho})(t)\|_{L^q} = \|\rho_0 - \tilde{\rho}\|_{L^q} \quad \text{for } \forall t \in [0, T]. \quad (3.3)$$

In view of (1.1)–(1.4), we have the following standard energy estimates.

Lemma 3.1 *For every $0 < T < T^*$, one has*

$$\begin{aligned} & \sup_{0 \leq t \leq T} \int \left(|\rho^{1/2} u|^2 + |\nabla d|^2 \right) dx + 2 \int_0^T \int (|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2) dx dt \\ & \leq \int \left(|\rho^{1/2} u|^2 + |\nabla d|^2 \right) (x, 0) dx \triangleq E_0. \end{aligned} \quad (3.4)$$

Proof. Multiplying (1.2) by u in L^2 and integrating by parts, by (1.3) we know that

$$\frac{1}{2} \frac{d}{dt} \int \rho |u|^2 dx + \int_\Omega |\nabla u|^2 dx = - \int (u \cdot \nabla d \cdot \Delta d) dx. \quad (3.5)$$

Due to the fact that $|d| = 1$, multiplying (1.4) by $(\Delta d + |\nabla d|^2 d)$ in L^2 , we obtain after integrating the resulting equations by parts over \mathbb{R}^2 that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int |\nabla d|^2 dx + \int_\Omega |\Delta d + |\nabla d|^2 d|^2 dx \\ & = \int (u \cdot \nabla d \cdot \Delta d) dx + \int (|\nabla d|^2 d \cdot d_t + |\nabla d|^2 u \cdot \nabla d \cdot d) dx \\ & = \int (u \cdot \nabla d \cdot \Delta d) dx + \frac{1}{2} \int (|\nabla d|^2 \partial_t |d|^2 + |\nabla d|^2 u \cdot \nabla |d|^2) dx \\ & = \int (u \cdot \nabla d \cdot \Delta d) dx, \end{aligned} \quad (3.6)$$

which, combined with (3.5), immediately leads to (3.4). \square

To be continued, we need the following key estimates on $\|\nabla^2 d\|_{L^2(0, T; L^2)}$.

Lemma 3.2 *Assume that the initial data satisfies*

$$\exp \left(2 \left(\|\rho_0^{1/2} u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2 \right) \right) \|\nabla d_0\|_{L^2}^2 \leq \frac{1}{16}, \quad (3.7)$$

then it holds for every $T \in (0, T^)$ that*

$$\sup_{0 \leq t \leq T} \|\nabla d\|_{L^2}^2 + \int_0^T \|\nabla^2 d\|_{L^2}^2 dt \leq \frac{1}{16}. \quad (3.8)$$

Proof. After integrating by parts, we easily deduce from the identity $|d| = 1$ that

$$\begin{aligned} \int |\Delta d + |\nabla d|^2 d|^2 dx &= \int (|\Delta d|^2 + |\nabla d|^4) dx - 2 \int |\nabla d|^2 (d \cdot \Delta d) dx \\ &= \int (|\Delta d|^2 - |\nabla d|^4) dx. \end{aligned} \quad (3.9)$$

On the other hand, integration by parts, together with the divergence-free condition (1.3), gives

$$\begin{aligned} \int (u \cdot \nabla d \cdot \Delta d) dx &= - \int \left(\partial_j u^i \partial_i d^k \partial_j d^k + u^i \partial_{ij}^2 d^k \partial_j d^k \right) dx \\ &= - \int \left(\partial_j u^i \partial_i d^k \partial_j d^k \right) dx \leq \|\nabla u\|_{L^2} \|\nabla d\|_{L^4}^2, \end{aligned} \quad (3.10)$$

where and in what follows the repeated indices denotes the summation over the indices.

Putting (3.9), (3.10) into (3.6) and recalling the fact that

$$\|\Delta d\|_{L^2}^2 = \|\nabla^2 d\|_{L^2}^2,$$

we obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla d\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 \leq \|\nabla u\|_{L^2} \|\nabla d\|_{L^4}^2 + \|\nabla d\|_{L^4}^4,$$

which, combined with (2.1) and the Cauchy-Schwarz inequality, yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla d\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 &\leq \sqrt{2} \|\nabla u\|_{L^2} \|\nabla d\|_{L^2} \|\nabla^2 d\|_{L^2} + 2 \|\nabla d\|_{L^2}^2 \|\nabla^2 d\|_{L^2}^2 \\ &\leq \left(2 \|\nabla d\|_{L^2}^2 + \frac{1}{4} \right) \|\nabla^2 d\|_{L^2}^2 + 2 \|\nabla u\|_{L^2}^2 \|\nabla d\|_{L^2}^2. \end{aligned} \quad (3.11)$$

It follows from (3.7) that

$$\|\nabla d_0\|_{L^2}^2 \leq e^{2E_0} \|\nabla d_0\|_{L^2}^2 \leq \frac{1}{16},$$

and thus, by the local existence theorem and the continuity argument we see that there exists a $T_1 > 0$ such that for any $t \in [0, T_1]$,

$$\|\nabla d\|_{L^2}^2 \leq \frac{1}{8}. \quad (3.12)$$

Set

$$\tilde{T} \triangleq \sup\{T \mid (3.12) \text{ holds}\}.$$

Then it follows from (3.11)–(3.12) that for any $t \in [0, \tilde{T}]$,

$$\frac{d}{dt} \|\nabla d\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 \leq 4 \|\nabla u\|_{L^2}^2 \|\nabla d\|_{L^2}^2,$$

which, together with Gronwall's inequality and (3.4), leads to

$$\|\nabla d\|_{L^2}^2 + \int_0^t \|\nabla^2 d\|_{L^2}^2 d\tau \leq \exp \left(4 \int_0^t \|\nabla u\|_{L^2}^2 d\tau \right) \|\nabla d_0\|_{L^2}^2$$

$$\leq e^{2E_0} \|\nabla d_0\|_{L^2}^2 \leq \frac{1}{16}. \quad (3.13)$$

Combining (3.4), (3.13) with the continuity argument immediately implies that (3.12) holds for all $0 < \tilde{T} < T^*$, and thus, the proof of (3.8) is finished. \square

By Lemmas 2.4 and 3.2, we can now derive the estimates of $\|\nabla u\|_{L^2}$ and $\|\nabla^2 d\|_{L^2}$ which is the most important step among the proofs.

Lemma 3.3 *For every $0 < T < T^*$, one has*

$$\begin{aligned} & \sup_{0 < t \leq T} (\|u\|_{H^1}^2 + \|\nabla d\|_{H^1}^2 + \|d_t\|_{L^2}^2) \\ & + \int_0^T \left(\|\nabla^2 u\|_{L^2}^2 + \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|d_t\|_{H^1}^2 + \|\nabla d\|_{H^2}^2 \right) dt \leq C, \end{aligned} \quad (3.14)$$

which particularly gives

$$\int_0^T \|\rho^{1/2} u_t\|_{L^2}^2 \leq C. \quad (3.15)$$

Proof. Let $\dot{f} \triangleq f_t + u \cdot \nabla f$ denote the material derivative. Also set

$$M(d) \triangleq \nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2 \mathbb{I}_2, \quad (\nabla d \otimes \nabla d)_{ij} \triangleq \frac{\partial d}{\partial x_i} \cdot \frac{\partial d}{\partial x_j}, \quad 1 \leq i, j \leq 2,$$

then it is easily seen that

$$\nabla d \cdot \Delta d = \operatorname{div}(M(d)).$$

To prove (3.14), multiplying (1.2) by u_t and integrating it by parts over \mathbb{R}^2 , we deduce

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|\rho^{1/2} \dot{u}\|_{L^2}^2 \\ & = - \int \operatorname{div} M(d) \cdot u_t dx + \int \rho u \cdot \nabla u \cdot \dot{u} dx \\ & = \frac{d}{dt} \int M(d) : \nabla u dx - \int M(d)_t : \nabla u dx + \int \rho u \cdot \nabla u \cdot \dot{u} dx \\ & \leq \frac{d}{dt} \int M(d) : \nabla u dx + \frac{1}{2} \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \frac{1}{4} \|\nabla d_t\|_{L^2}^2 + C(\|u\|_{L^\infty}^2 + \|\nabla d\|_{L^\infty}^2) \|\nabla u\|_{L^2}^2, \end{aligned}$$

where we have also used (3.2) and Cauchy-Schwarz inequality. As a result,

$$\begin{aligned} & \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|\rho^{1/2} \dot{u}\|_{L^2}^2 \\ & \leq 2 \frac{d}{dt} \int M(d) : \nabla u dx + \frac{1}{2} \|\nabla d_t\|_{L^2}^2 + C(\|u\|_{L^\infty}^2 + \|\nabla d\|_{L^\infty}^2) \|\nabla u\|_{L^2}^2. \end{aligned} \quad (3.16)$$

Next, one easily obtains from (1.4) that

$$\begin{aligned} & \frac{d}{dt} \|\nabla d\|_{L^2}^2 + (\|d_t\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) \leq C \int (|u|^2 |\nabla d|^2 + |\nabla d|^4) dx \\ & \leq C (\|u\|_{L^\infty}^2 + \|\nabla d\|_{L^\infty}^2) \|\nabla d\|_{L^2}^2. \end{aligned} \quad (3.17)$$

To deal with the term $\|\nabla d_t\|_{L^2}^2$ on the right-hand side of (3.16), we first apply ∇ to both sides of (1.4) to get that

$$\nabla d_t - \nabla \Delta d = -\nabla(u \cdot \nabla d) + \nabla(|\nabla d|^2 d), \quad (3.18)$$

from which it follows that

$$\begin{aligned} & \frac{d}{dt} \|\nabla^2 d\|_{L^2}^2 + (\|\nabla d_t\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2) \\ & \leq \int (|\nabla(|\nabla d|^2 d)|^2 + |\nabla(u \cdot \nabla d)|^2) dx \\ & \leq C \int (|\nabla d|^6 + |\nabla d|^2 |\nabla^2 d|^2 + |\nabla u|^2 |\nabla d|^2 + |u|^2 |\nabla^2 d|^2) dx \\ & \leq C (\|u\|_{L^\infty}^2 + \|\nabla d\|_{L^\infty}^2) (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2), \end{aligned} \quad (3.19)$$

where we have used (2.1) and (3.4) to get that

$$\|\nabla d\|_{L^6}^6 \leq C \|\nabla d\|_{L^\infty}^2 \|\nabla d\|_{L^2}^2 \|\nabla^2 d\|_{L^2}^2 \leq C \|\nabla d\|_{L^\infty}^2 \|\nabla^2 d\|_{L^2}^2.$$

Using (2.1) and (3.4) again, we have

$$\int M(d) : \nabla u dx \leq \frac{1}{4} \|\nabla u\|_{L^2}^2 + C \|\nabla d\|_{L^2}^2 \|\nabla^2 d\|_{L^2}^2 \leq \frac{1}{4} \|\nabla u\|_{L^2}^2 + C_1 \|\nabla^2 d\|_{L^2}^2.$$

Taking this into account, multiplying (3.19) by $2C_1 + 1$, and adding the resulting inequality, (3.16) and (3.17) together, we obtain after integrating the resulting inequality over (s, t) with $0 \leq s < t < T$ that

$$\begin{aligned} & (\|\nabla u\|_{L^2}^2 + \|\nabla d\|_{H^1}^2)(t) + \int_s^t \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|d_t\|_{H^1}^2 + \|\nabla^2 d\|_{H^1}^2 \right) d\tau \\ & \leq C (\|\nabla u\|_{L^2}^2 + \|\nabla d\|_{H^1}^2)(s) + C \int_s^t (\|u\|_{L^\infty}^2 + \|\nabla d\|_{L^\infty}^2) (\|\nabla u\|_{L^2}^2 + \|\nabla d\|_{H^1}^2) d\tau, \end{aligned}$$

and consequently,

$$\begin{aligned} & (\|\nabla u\|_{L^2}^2 + \|\nabla d\|_{H^1}^2)(t) + \int_s^t \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|d_t\|_{H^1}^2 + \|\nabla^2 d\|_{H^1}^2 \right) d\tau \\ & \leq C (\|\nabla u\|_{L^2}^2 + \|\nabla d\|_{H^1}^2)(s) \exp \left(C \int_s^t (\|u\|_{L^\infty}^2 + \|\nabla d\|_{L^\infty}^2) d\tau \right). \end{aligned} \quad (3.20)$$

Clearly, it remains to estimate $\|(u, \nabla d)\|_{L^\infty}$. To this end, let

$$\Phi(t) \triangleq e + \sup_{0 \leq \tau \leq t} (\|\nabla u\|_{L^2}^2 + \|\nabla d\|_{H^1}^2)(\tau) + \int_0^t \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|d_t\|_{H^1}^2 + \|\nabla^2 d\|_{H^1}^2 \right) d\tau.$$

First, in view of (2.4) and (3.2)–(3.4), we have

$$\|u\|_{L^2}^2 \leq C \left(\|\rho^{1/2} u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right) \leq C (1 + \|\nabla u\|_{L^2}^2). \quad (3.21)$$

Next, using Lemma 2.3, (2.1), (3.2) and (3.4), we deduce from Hölder and Cauchy-Schwarz inequalities that

$$\|\nabla^2 u\|_{L^2} \leq C (\|\rho \dot{u}\|_{L^2} + \|\nabla d \cdot \Delta d\|_{L^2})$$

$$\leq C \left(\|\rho^{1/2}\dot{u}\|_{L^2} + \|\nabla^2 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2} \right), \quad (3.22)$$

which, combined with (3.8), yields

$$\begin{aligned} \int_s^t \|\nabla^2 u\|_{L^2}^2 d\tau &\leq C \int_s^t \left(\|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^4 + \|\nabla^3 d\|_{L^2}^2 \right) d\tau \\ &\leq C \sup_{s \leq \tau \leq t} \|\nabla^2 d\|_{L^2}^2 + C \int_s^t \left(\|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 \right) d\tau. \end{aligned} \quad (3.23)$$

Thus, recalling the definition of $\Phi(T)$ and using (3.4), (3.21) and (3.23), we infer from Lemma 2.4 that for any $0 \leq s < t \leq T < T^*$,

$$\begin{aligned} \|u\|_{L^2(s,t;L^\infty)}^2 &\leq C \left(1 + \|u\|_{L^2(s,t;H^1)}^2 \ln(e + \|u\|_{L^2(s,t;W^{1,4})}) \right) \\ &\leq C \left(1 + \|\nabla u\|_{L^2(s,t;L^2)}^2 \ln(e + \|u\|_{L^2(s,t;H^2)}) \right) \\ &\leq C \left(1 + \|\nabla u\|_{L^2(s,t;L^2)}^2 \ln(e + \|\nabla^2 u\|_{L^2(s,t;L^2)}) \right) \\ &\leq C \left(1 + \|\nabla u\|_{L^2(s,t;L^2)}^2 \ln(C\Phi(t)) \right). \end{aligned} \quad (3.24)$$

In a similar manner, by (2.6) and (3.4) one has

$$\begin{aligned} \|\nabla d\|_{L^2(s,t;L^\infty)}^2 &\leq C \left(1 + \|\nabla d\|_{L^2(s,t;H^1)}^2 \ln(e + \|\nabla d\|_{L^2(s,t;W^{1,4})}) \right) \\ &\leq C \left(1 + \|\nabla^2 d\|_{L^2(s,t;L^2)}^2 \ln(e + \|\nabla^2 d\|_{L^2(s,t;H^1)}) \right) \\ &\leq C \left(1 + \|\nabla^2 d\|_{L^2(s,t;L^2)}^2 \ln(C\Phi(t)) \right). \end{aligned} \quad (3.25)$$

For any $0 \leq s < t \leq T < T^*$, putting (3.24) and (3.25) into (3.20) gives

$$\begin{aligned} \Phi(t) &\leq C\Phi(s) \exp \left\{ C_2 \left(\|\nabla u\|_{L^2(s,t;L^2)}^2 + \|\nabla^2 d\|_{L^2(s,t;L^2)}^2 \right) \ln(C_1\Phi(t)) \right\} \\ &\leq C\Phi(s) [C_1\Phi(t)]^{C_2 \left(\|\nabla u\|_{L^2(s,t;L^2)}^2 + \|\nabla^2 d\|_{L^2(s,t;L^2)}^2 \right)}. \end{aligned} \quad (3.26)$$

It follows from (3.4) and (3.8) that there exists a positive constant $\delta > 0$ such that

$$C_2 \left(\|\nabla u\|_{L^2(T-\delta,T;L^2)}^2 + \|\nabla^2 d\|_{L^2(T-\delta,T;L^2)}^2 \right) \leq \frac{1}{2},$$

which, inserted into (3.26), leads to

$$\Phi(T) \leq C\Phi(T-\delta) [C_1\Phi(T)]^{1/2} \leq \frac{1}{2}\Phi(T) + C\Phi^2(T-\delta),$$

so that

$$\Phi(T) \leq C(T)\Phi^2(T-\delta). \quad (3.27)$$

As a result of (3.27), we see that $\Phi(T)$ is bounded for any $0 < T < T^*$ since the local existence theorem indicates $\Phi(T-\delta) < \infty$ for any $0 < T < T^*$. This, together with (3.21) and (3.22), finishes the proof of (3.14).

Furthermore, recalling the definition of material derivative (i.e. “ \cdot ”), one gets from (2.1), (3.2) and (3.14) that

$$\begin{aligned}
\int_0^T \|\rho^{1/2} u_t\|_{L^2}^2 dt &\leq \int_0^T \|\rho^{1/2} \dot{u}\|_{L^2}^2 dt + \int_0^T \int \rho |u|^2 |\nabla u|^2 dx dt \\
&\leq C + C \int_0^T \|u\|_{L^4}^2 \|\nabla u\|_{L^4}^2 dt \\
&\leq C + C \int_0^T \|u\|_{L^2} \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2} dt \\
&\leq C + C \int_0^T \|\nabla^2 u\|_{L^2}^2 dt \leq C,
\end{aligned}$$

which immediately proves (3.15). The proof of Lemma 3.3 is therefore complete. \square

Next, we proceed to estimate $\|\rho^{1/2} u_t\|_{L^2}$ and $\|\nabla d_t\|_{L^2}$.

Lemma 3.4 *For every $0 < T < T^*$, one has*

$$\sup_{0 < t \leq T} \left(\|\rho^{1/2} u_t\|_{L^2}^2 + \|d_t\|_{H^1}^2 \right) + \int_0^T (\|\nabla u_t\|_{L^2}^2 + \|d_{tt}\|_{L^2}^2 + \|\nabla^2 d_t\|_{L^2}^2) dt \leq C, \quad (3.28)$$

and moreover,

$$\sup_{0 < t \leq T} (\|u\|_{H^2}^2 + \|\nabla d\|_{H^2}^2) + \int_0^T (\|\nabla u\|_{W^{1,4}}^2 + \|\nabla^2 d\|_{H^2}^2) dt \leq C. \quad (3.29)$$

Proof. Differentiating (1.2) with respect to t gives

$$\rho u_{tt} + \rho u \cdot \nabla u_t - \Delta u_t = -\rho_t(u_t + u \cdot \nabla u) - \rho u_t \cdot \nabla u - \nabla P_t - \operatorname{div} M_t,$$

which, multiplied by u_t in L^2 and integrated by parts over \mathbb{R}^2 , results in

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|\rho^{1/2} u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \\
&= - \int \rho_t(u_t + u \cdot \nabla u) \cdot u_t dx - \int \rho u_t \cdot \nabla u \cdot u_t dx + \int M_t : \nabla u_t dx \\
&\triangleq I_1 + I_2 + I_3
\end{aligned} \quad (3.30)$$

where $A : B = \sum_{i,j=1}^2 a_{ij} b_{ij}$ for $A = (a_{ij})_{2 \times 2}$ and $B = (b_{ij})_{2 \times 2}$.

We are now in a position of estimating the right-hand side of (3.30) term by term. First, using (1.1) and integrating by parts, by Lemma 2.1, (3.2) and (3.14) we deduce

$$\begin{aligned}
I_1 &= \int (\rho u \cdot \nabla |u_t|^2 + \rho u \cdot \nabla (u \cdot \nabla u \cdot u_t)) dx \\
&\leq C \int (\rho |u| |u_t| |\nabla u_t| + \rho |u| |\nabla u|^2 |u_t| + \rho |u|^2 |\nabla^2 u| |u_t| + \rho |u|^2 |\nabla u| |\nabla u_t|) dx \\
&\leq C \left(\|u\|_{L^\infty} \|\rho^{1/2} u_t\|_{L^2} \|\nabla u_t\|_{L^2} + \|u\|_{L^\infty} \|\nabla u\|_{L^4}^2 \|\rho^{1/2} u_t\|_{L^2} \right)
\end{aligned}$$

$$\begin{aligned}
& +C \left(\|u\|_{L^\infty}^2 \|\rho^{1/2} u_t\|_{L^2} \|\nabla^2 u\|_{L^2} + \|u\|_{L^\infty}^2 \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2} \right) \\
& \leq \frac{1}{4} \|\nabla u_t\|_{L^2}^2 + C \left(1 + \|\rho^{1/2} u_t\|_{L^2}^4 + \|\nabla^2 u\|_{L^2}^2 \right),
\end{aligned}$$

where we have used Cauchy-Schwarz inequality and the following estimate due to Lemma 2.1 and (3.14):

$$\|u\|_{L^\infty} \leq C \|u\|_{W^{1,4}} \leq C \left(\|u\|_{H^1} + \|\nabla u\|_{L^2}^{1/2} \|\nabla^2 u\|_{L^2}^{1/2} \right) \leq C \left(1 + \|\nabla^2 u\|_{L^2}^{1/2} \right). \quad (3.31)$$

Due to (2.4), (3.2) and (3.3), we have

$$\|u_t\|_{L^2}^2 \leq C \left(\|\rho^{1/2} u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \right), \quad (3.32)$$

and thus, by (2.1), (3.2) and (3.14) the second term I_2 can be bounded as follows:

$$\begin{aligned}
|I_2| & \leq C \int \rho |u_t|^2 |\nabla u| dx \\
& \leq C \|\nabla u\|_{L^4} \|u_t\|_{L^4} \|\rho^{1/2} u_t\|_{L^2} \\
& \leq C \|\nabla u\|_{L^2}^{1/2} \|\nabla^2 u\|_{L^2}^{1/2} \|u_t\|_{L^2}^{1/2} \|\nabla u_t\|_{L^2}^{1/2} \|\rho^{1/2} u_t\|_{L^2} \\
& \leq C \left(\|\nabla u_t\|_{L^2} + \|\rho^{1/2} u_t\|_{L^2} \right) \|\nabla^2 u\|_{L^2}^{1/2} \|\rho^{1/2} u_t\|_{L^2} \\
& \leq \frac{1}{4} \|\nabla u_t\|_{L^2}^2 + C \left(1 + \|\nabla^2 u\|_{L^2}^2 + \|\rho^{1/2} u_t\|_{L^2}^4 \right).
\end{aligned}$$

Finally, it is easily seen from (2.3) and (3.14) that

$$\begin{aligned}
|I_3| & \leq C \|\nabla d\|_{L^\infty} \|\nabla d_t\|_{L^2} \|\nabla u_t\|_{L^2} \\
& \leq \frac{1}{4} \|\nabla u_t\|_{L^2}^2 + C \left(1 + \|\nabla^3 d\|_{L^2}^2 \right) \|\nabla d_t\|_{L^2}^2.
\end{aligned}$$

Substituting the estimates of I_1, I_2 and I_3 into (3.30), one obtains

$$\begin{aligned}
& \frac{d}{dt} \|\rho^{1/2} u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \\
& \leq C \left(1 + \|\nabla^2 u\|_{L^2}^2 + \|\rho^{1/2} u_t\|_{L^2}^4 \right) + C \left(1 + \|\nabla d\|_{H^2}^2 \right) \|\nabla d_t\|_{L^2}^2.
\end{aligned} \quad (3.33)$$

To estimate $\|\nabla d_t\|_{L^2}$, we differentiate (1.4) with respect to t to get

$$d_{tt} - \Delta d_t = (|\nabla d|^2 d - u \cdot \nabla d)_t,$$

and hence, using Lemma 2.1 and (3.14), we deduce after direct calculations that

$$\begin{aligned}
& \frac{d}{dt} \|\nabla d_t\|_{L^2}^2 + (\|d_{tt}\|_{L^2}^2 + \|\Delta d_t\|_{L^2}^2) \\
& \leq C \int (|\nabla d|^2 |\nabla d_t|^2 + |\nabla d|^4 |d_t|^2 + |u_t|^2 |\nabla d|^2 + |u|^2 |\nabla d_t|^2) dx \\
& \leq C (\|\nabla d\|_{L^\infty}^2 + \|u\|_{L^\infty}^2) \|\nabla d_t\|_{L^2}^2 + C \|\nabla d\|_{L^\infty}^4 \|d_t\|_{L^2}^2 + C \|\nabla d\|_{L^4}^2 \|u_t\|_{L^4}^2 \\
& \leq C (\|\nabla d\|_{H^2}^2 + \|u\|_{H^2}^2) \|\nabla d_t\|_{L^2}^2 + C_1 \left(\|\nabla d\|_{H^2}^2 + \|\rho^{1/2} u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \right),
\end{aligned} \quad (3.34)$$

where we have also used (3.32) and the following estimate due to (2.1), (2.3) and (3.14):

$$\|\nabla d\|_{L^\infty}^4 \leq C (\|\nabla d\|_{L^4}^4 + \|\nabla^2 d\|_{L^4}^4) \leq C (1 + \|\nabla^2 d\|_{L^2}^2 \|\nabla^3 d\|_{L^2}^2) \leq C (1 + \|\nabla^3 d\|_{L^2}^2).$$

Now, multiplying (3.33) by $2C_1 + 1$ and adding it to (3.34), we see that

$$\begin{aligned} & \frac{d}{dt} \left(\|\rho^{1/2} u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 \right) + (\|\nabla u_t\|_{L^2}^2 + \|\Delta d_t\|_{L^2}^2 + \|d_{tt}\|_{L^2}^2) \\ & \leq C (1 + \|\nabla d\|_{H^2}^2 + \|u\|_{H^2}^2) (1 + \|\nabla d_t\|_{L^2}^2) + C \|\rho^{1/2} u_t\|_{L^2}^4, \end{aligned}$$

which, combined with (3.14), (3.15) and Gronwall's inequality, leads to (3.28), since the compatibility condition stated in (1.13)₂ implies that $(\rho^{1/2} u_t)(x, 0) \in L^2(\mathbb{R}^2)$ is well defined.

Using (2.1), (3.2), (3.14) and (3.28), we have by (2.5) that

$$\begin{aligned} \|\nabla^2 u\|_{L^2} & \leq C (\|\rho u_t\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2} + \|\nabla d \cdot \Delta d\|_{L^2}) \\ & \leq C \left(\|\rho^{1/2} u_t\|_{L^2} + \|u\|_{L^4} \|\nabla u\|_{L^4} + \|\nabla d\|_{L^4} \|\nabla^2 d\|_{L^4} \right) \\ & \leq C \left(1 + \|\nabla^2 u\|_{L^2}^{1/2} + \|\nabla^3 d\|_{L^2}^{1/2} \right). \end{aligned} \tag{3.35}$$

Similarly, one also infers from (3.18) that

$$\begin{aligned} \|\nabla^3 d\|_{L^2} & \leq C (\|\nabla d_t\|_{L^2} + \|\nabla(u \cdot \nabla d)\|_{L^2} + \|\nabla(|\nabla d|^2 d)\|_{L^2}) \\ & \leq C (1 + \|u\|_{L^4} \|\nabla^2 d\|_{L^4} + \|\nabla u\|_{L^4} \|\nabla d\|_{L^4} + \|\nabla d\|_{L^6}^3 + \|\nabla d\|_{L^4} \|\nabla^2 d\|_{L^4}) \\ & \leq C \left(1 + \|\nabla^2 u\|_{L^2}^{1/2} + \|\nabla^3 d\|_{L^2}^{1/2} \right), \end{aligned}$$

from which, (3.35) and Young's inequality, we arrive at

$$\sup_{0 \leq t \leq T} (\|u\|_{H^2} + \|\nabla d\|_{H^2}) \leq C. \tag{3.36}$$

Using (2.5) and Lemma 2.1 again, we deduce from (3.2), (3.28) and (3.36) that

$$\begin{aligned} \int_0^T \|\nabla^2 u\|_{L^4}^2 dt & \leq C \int_0^T (\|\rho u_t\|_{L^4}^2 + \|\rho u \cdot \nabla u\|_{L^4}^2 + \|\nabla d \cdot \Delta d\|_{L^4}^2) dt \\ & \leq C \int_0^T \left(\|\rho^{1/2} u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + \|u\|_{H^2}^4 + \|\nabla d\|_{H^2}^4 \right) dt \\ & \leq C + C \int_0^T \|\nabla u_t\|_{L^2}^2 dt \leq C. \end{aligned} \tag{3.37}$$

Moreover, by virtue Lemma 2.1, (3.28) and (3.29) we infer from (1.4) that

$$\begin{aligned} \int_0^T \|\nabla^2 d\|_{H^2}^2 dt & \leq C \int_0^T (\|d_t\|_{H^2}^2 + \|u \cdot \nabla d\|_{H^2}^2 + \| |\nabla d|^2 d \|_{H^2}^2) dt \\ & \leq C \int_0^t (\|d_t\|_{H^2}^2 + \|u\|_{H^2}^2 \|\nabla d\|_{H^2}^2 + \|\nabla d\|_{H^2} + \|\nabla d\|_{H^2}^6) dt \\ & \leq C, \end{aligned}$$

where we have used the following Moser's type calculus inequality (see [30]) that for $f, g \in H^s(\mathbb{R}^2)$ with $s \geq 2$,

$$\|fg\|_{H^s} \leq C (\|f\|_{L^\infty} \|g\|_{H^s} + \|f\|_{H^s} \|g\|_{L^\infty}) \leq C \|f\|_{H^s} \|g\|_{H^s}.$$

This, together with (3.36) and (3.37), leads to (3.29) immediately. \square

The last step is to estimate the first and second order derivatives of the density.

Lemma 3.5 *For every $0 < T < T^*$, one has*

$$\sup_{0 \leq t \leq T} (\|\nabla \rho\|_{H^1} + \|\rho_t\|_{H^1}) + \int_0^T \|\nabla u\|_{H^2}^2 dt \leq C. \quad (3.38)$$

Proof. Differentiating (1.1) with respect to x_i ($i = 1, 2$), multiplying the resulting equation by $|\nabla \rho|^{p-2} \partial_i \rho$ with $p \geq 2$, and integrating it by parts over \mathbb{R}^2 , we obtain after summing up that

$$\frac{d}{dt} \|\nabla \rho\|_{L^p}^p \leq C \|\nabla u\|_{L^\infty} \|\nabla \rho\|_{L^p}^p \leq C \|\nabla u\|_{W^{1,4}} \|\nabla \rho\|_{L^p}^p,$$

which, combined with (3.29) and Gronwall's inequality, yields

$$\|\nabla \rho\|_{L^p}^p \leq C \|\nabla \rho_0\|_{L^p}^p \exp \left(C \int_0^T \|\nabla u\|_{W^{1,4}} dt \right) \leq C, \quad \forall p \geq 2. \quad (3.39)$$

Similarly, by (3.38) we also deduce from (1.1) that

$$\begin{aligned} \frac{d}{dt} \|\nabla^2 \rho\|_{L^2}^2 &\leq C \|\nabla u\|_{L^\infty} \|\nabla^2 \rho\|_{L^2}^2 + C \|\nabla^2 u\|_{L^4} \|\nabla \rho\|_{L^4} \|\nabla^2 \rho\|_{L^2} \\ &\leq C \|\nabla u\|_{W^{1,4}} (1 + \|\nabla^2 \rho\|_{L^2}^2), \end{aligned}$$

so that

$$\|\nabla^2 \rho\|_{L^2}^2 \leq C (1 + \|\nabla^2 \rho_0\|_{L^2}^2) \exp \left(C \int_0^T \|\nabla u\|_{W^{1,4}} dt \right) \leq C. \quad (3.40)$$

As a result of (3.28), (3.39) and (3.40), one easily gets from (1.1) that $\|\rho_t\|_{H^1} \leq C$.

Finally, it follows from (2.5), (3.2), (3.14), (3.28) and (3.38) that

$$\begin{aligned} \|\nabla u\|_{H^2} &\leq C (\|\rho u_t\|_{H^1} + \|\rho u \cdot \nabla u\|_{H^1} + \|\nabla d \cdot \Delta d\|_{H^1}) \\ &\leq C \left(\|\rho^{1/2} u_t\|_{L^2} + \|\nabla \rho\|_{L^4} \|u_t\|_{L^4} + \|\nabla u_t\|_{L^2} \right) \\ &\quad + C (\|u\|_{H^2}^2 + \|u\|_{L^\infty} \|\nabla \rho\|_{L^4} \|\nabla u\|_{L^4} + \|\nabla d\|_{H^2}^2) \\ &\leq C (1 + \|\nabla u_t\|_{L^2}), \end{aligned}$$

where we have also used Lemmas 2.1 and 2.2. Consequently,

$$\int_0^T \|\nabla u\|_{H^2}^2 dt \leq C + C \int_0^T \|\nabla u_t\|_{L^2}^2 dt \leq C,$$

which, together with (3.39) and (3.40), proves (3.38). \square

Collecting all the estimates in (3.2), (3.3) and Lemmas 3.1–3.5 together, we arrive at (3.1), and hence, the proof of Theorem 1.1 is complete.

4 Proofs of Theorems 1.2 and 1.3

This section is concerned with the proofs of Theorems 1.2 and 1.3. We first prove Theorem 1.2 by using contradiction arguments. So, to do this, we assume otherwise that

$$\lim_{T \rightarrow T^*} \int_0^T \|\nabla d\|_{L^r}^s dt \leq M_0 < \infty \quad (4.1)$$

with any (r, s) satisfying (1.17).

We begin the proof with the observation from the proof of Theorem 1.1 that, to remove the smallness condition (1.14) and to obtain a global strong solution with generally large initial data, it suffices to achieve the estimate of $\|\nabla^2 d\|_{L^2(0,T;L^2)}$ for any $0 < T < T^*$. Moreover, it follows from (3.9) and (3.4) that for any $0 < T < T^*$,

$$\begin{aligned} \int_0^T \|\nabla^2 d\|_{L^2}^2 dt &= \int_0^T \|\Delta d + |\nabla d|^2 d\|_{L^2}^2 dt + \int_0^T \|\nabla d\|_{L^4}^4 dt \\ &\leq \frac{1}{2} E_0 + \int_0^T \|\nabla d\|_{L^4}^4 dt. \end{aligned} \quad (4.2)$$

Therefore, to bound $\|\nabla^2 d\|_{L^2(0,T;L^2)}$, we only need to deal with $\|\nabla d\|_{L^4(0,T;L^4)}$. This will be done in the following.

On one hand, assume that (r, s) satisfies

$$\frac{1}{r} + \frac{1}{s} \leq \frac{1}{2} \quad \text{with} \quad 4 \leq r \leq \infty. \quad (4.3)$$

Then, using Hölder inequality, (2.1) and (3.4), we find

$$\begin{aligned} \|\nabla d\|_{L^4}^4 &= \|\nabla d\|_{L^4}^2 \|\nabla d\|_{L^4}^2 \\ &\leq C \|\nabla d\|_{L^2}^{(r-4)/(r-2)} \|\nabla d\|_{L^r}^{r/(r-2)} \|\nabla d\|_{L^2} \|\nabla^2 d\|_{L^2} \\ &\leq C \|\nabla d\|_{L^r}^{r/(r-2)} \|\nabla^2 d\|_{L^2} \\ &\leq \frac{1}{2} \|\nabla^2 d\|_{L^2}^2 + C(1 + \|\nabla d\|_{L^r}^s). \end{aligned} \quad (4.4)$$

Thus, putting (4.4) into (4.2) and using (4.1), we obtain

$$\int_0^T \|\nabla^2 d\|_{L^2}^2 dt \leq E_0 + C \int_0^T (1 + \|\nabla d\|_{L^r}^s) dt \leq C, \quad (4.5)$$

provided (r, s) satisfies (4.3).

On the other hand, assume that (r, s) satisfies

$$\frac{1}{r} + \frac{1}{s} \leq \frac{1}{2} \quad \text{with} \quad 2 < r < 4. \quad (4.6)$$

Then, by virtue of (2.2), (3.4) and Hölder inequality we find that

$$\begin{aligned} \|\nabla d\|_{L^4} &\leq C \|\nabla d\|_{L^r}^{1-\alpha} \|\nabla d\|_{L^p}^\alpha \\ &\leq C \|\nabla d\|_{L^r}^{1-\alpha} \|\nabla d\|_{L^2}^{2\alpha/p} \|\nabla^2 d\|_{L^2}^{\alpha(p-2)/p} \\ &\leq C \|\nabla d\|_{L^r}^{1-\alpha} \|\nabla^2 d\|_{L^2}^{\alpha(p-2)/p} \end{aligned} \quad (4.7)$$

with

$$\frac{2r}{r-2} < p < \infty, \quad \alpha = \frac{(4-r)p}{4(p-r)} \in \left(0, \frac{1}{2}\right).$$

As a result of (4.7), we have by Young's inequality that

$$\|\nabla d\|_{L^4}^4 \leq C \|\nabla d\|_{L^r}^{4(1-\alpha)} \|\nabla^2 d\|_{L^2}^{4\alpha(p-2)/p}$$

$$\begin{aligned}
&\leq \frac{1}{2} \|\nabla^2 d\|_{L^2}^2 + C \|\nabla d\|_{L^r}^{4(1-\alpha)p/(p-2\alpha(p-2))} \\
&\leq \frac{1}{2} \|\nabla^2 d\|_{L^2}^2 + C (1 + \|\nabla d\|_{L^r}^s),
\end{aligned} \tag{4.8}$$

since direct calculations give

$$\frac{4(1-\alpha)p}{p-2\alpha(p-2)} = \frac{2r}{r-2} \leq s.$$

Thus, putting (4.8) into (4.2), by (4.1) we also obtain (4.5), provided (r, s) satisfies (4.6).

To conclude, we have proved that there exists a positive constant C , depending on the initial data, T^* and M_0 , such that for any $0 < T < T^*$,

$$\int_0^T \|\nabla^2 d\|_{L^2}^2 dt \leq C,$$

provided (4.1) holds. With the help of this and (3.2)–(3.4), following the arguments in the proofs of Lemmas 3.3–3.5, we arrive at (3.1), which, combined with the local existence theorem (see Lemma 2.5), implies the solutions can be extended beyond T^* . This immediately leads to a contradiction of T^* , and hence, the proof of Theorem 1.2 is complete. \square

Proof of Theorem 1.2. In fact, by applying the maximum principle to the equation of d_3 (i.e. the third component of d), we have

$$\inf_{x \in \mathbb{R}^2} d_3(x, t) \geq \inf_{x \in \mathbb{R}^2} d_{03} \geq \varepsilon, \quad \forall t > 0.$$

So, it follows from (3.4) and (1.19) that for any $0 < T < \infty$,

$$\int_0^T (\|\Delta d\|_{L^2}^2 + \|\nabla d\|_{L^4}^4) dt \leq C(E_0, \varepsilon) \int_0^T \|\Delta d + |\nabla d|^2 d\|_{L^2}^2 dt \leq C(E_0, \varepsilon). \tag{4.9}$$

This, combined with Theorem 1.2 with $r = s = 4$, implies that the strong solutions of (1.1)–(1.4), (1.10) and (1.11) exist for all $T > 0$. The proof of Theorem 1.3 is thus finished. \square

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